ON CURVES AND SURFACES OF $AW(k)$ TYPE

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ABSTRACT

In the present study we consider curves and surfaces of $AW(k)$ ($k=1, 2$ or $3$) type. We also give related examples of curves and surfaces satisfying $AW(k)$ type conditions.

Keywords: Frenet curve, curves and surfaces of $AW(k)$ type.

ÖZET

Bu çalışmada, $AW(k)$ ($k=1, 2$ yada $3$) tipinde eğri ve yüzeyler gözönüne alındı. $AW(k)$ şartını sağlayan eğri ve yüzeylere örnekler verildi.

Anahtar Kelimeler: Frenet eğrisi, $AW(k)$ tipinde eğri ve yüzey.

1- INTRODUCTION

Let $f : M \rightarrow \tilde{M}$ be an isometric immersion of an $n$-dimensional connected Riemannian manifold $M$ into an $m$-dimensional Riemannian manifold $\tilde{M}$. Letters $X, Y$ and $Z$ (resp. $\xi, \mu$ and $\tilde{\xi}$) vector fields tangent (resp. normal) to $M$. We denote the tangent bundle of $M$ (resp. $\tilde{M}$) by $TM$ (resp. $T\tilde{M}$), unit tangent bundle of $M$ by $UM$ and the normal bundle by $T_{\perp}M$. Let $\tilde{\nabla}$ and $\nabla$ be the Levi-Civita connections of $\tilde{M}$ and $M$, resp. Then the Gauss formula is given by

$$\tilde{\nabla}_XY = \nabla_X Y + h(X,Y)$$

(1)

where $h$ denotes the second fundamental form. The Weingarten formula is given by

$$\tilde{\nabla}_X \xi = -A_X \xi + D_X \xi$$

(2)

where $A$ denotes the shape operator and $D$ the normal connection. Clearly $h(X,Y) = h(Y,X)$ and $A$ is related to $h$ as $\{ A_X Y \} = \{ h(X,Y) \}$, where $\{ , \}$ denotes the Riemannian metrics of $M$ and $\tilde{M}$ [1].

Let $\{ e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_m \}$ be an local orthonormal frame field on $M$ where $\{ e_1, e_2, \ldots, e_n \}$ are tangent vector and $\{ e_{n+1}, \ldots, e_m \}$ are normal vector. The connection form $\omega_i^j$ are defined by

$$\tilde{\nabla}_{e_i} \omega_i^j = \sum_{\alpha} w_i^{\alpha} e_j ; \quad w_i^{\alpha} = -w_j^{\alpha}, \quad 1 \leq i, j \leq m$$

(3)

$$\nabla_{e_i} e_j = \sum_{k=1}^m w_j^k (e_i) e_k ,$$

(4)

$$D_{e_i} e_\alpha = \sum_{\beta=n+1}^m w_\alpha^\beta (e_i) e_\beta .$$

(5)

The covariant derivations of $h$ is defined by
\[ (\nabla h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \quad (6) \]

where \( X, Y, Z \) tangent vector fields over \( M \) and \( \nabla \) is the van der Waerden Bortolotti connection. Then we have

\[ (\nabla h)(Y, Z) = (\nabla h)(X, Z) = (\nabla h)(Y, X) \quad (7) \]

which is called codazzi equations.

If \( \nabla h = 0 \) then \( M \) is said to have parallel second fundamental form (i.e. \( I \)-parallel) [2].

It is well known that \( \nabla h \) is a normal bundle valued tensor of type \((0,3)\). We define the second covariant derivative of \( h \) by

\[ (\nabla h)(Y, Z) = h(\nabla h)(Y, Z) \quad (8) \]

For the orthonormal frame \( \{ e_1, e_2, ..., e_n \} \) of \( T_p M \) the mean curvature vector \( H \) of \( f \) is defined by

\[ H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \quad (9) \]

2. CURVES OF \( AW(k) \) TYPE

Let \( \gamma = \gamma(s) : I \subset IE \rightarrow IE^m \) be a unit speed curve in \( IE^m \). The curve \( \gamma \) is called Frenet curve of osculating order \( d \) if its higher order derivatives \( \gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(d)}(s) \) are linearly independent and \( \gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(d+1)}(s) \) are linearly dependent for all \( s \in I \). For each Frenet curve of order \( d \) one can associate an orthonormal \( d \)-frame \( v_1, v_2, ..., v_d \) along \( \gamma \) (such that \( T = \gamma'(s) = v_1 \)) called the Frenet frame and \( d \)-1 functions \( \kappa_1, \kappa_2, ..., \kappa_{d-1} : I \rightarrow IR \) called the Frenet curvatures, such that the Frenet formulas are defined in the usual way;

\[ T'(s) = v_1' = \kappa_1(s)v_1(s) \quad (10) \]

\[ v_2'(s) = -\kappa_1(s)T(s) + \kappa_2(s)v_2(s) \quad (11) \]

\[ v_3'(s) = -\kappa_{i-1}(s)v_{i-1}(s) + \kappa_i(s)v_i(s) \quad (12) \]

\[ v_{i+1}'(s) = -\kappa_i(s)v_i(s). \quad (13) \]

A regular curve \( \gamma = \gamma(s) : I \subset IE \rightarrow IE^m \) is called a \( W \)-curve of rank \( d \), if \( \gamma \) is a Frenet curve of osculating order \( d \) and the Frenet curvatures \( \kappa_i, 1 \leq i \leq d-1 \) are non zero constant and \( \kappa_d = 0 \). In particular, a \( W \)-curve \( \gamma(s) \) of rank 2 is called a geodesic circle. A \( W \)-curves of rank 3 is a right circular helix.

Let \( M \) be a smooth \( n \)-dimensional submanifold in \((n+d)\)-dimensional Euclidean space \( IE^{n+d} \). For \( x \in M \) and a unit vector \( X \in T_x M \), the vector \( X \) and the normal space \( N_{x} M \) determine a \((d+1)\)-dimensional affine subspace \( IE(x, X) \) of \( IE^{n+d} \). The intersection of \( M \) and \( IE(x, X) \) gives rise to a curve \( \gamma(s) \) (in a neighborhood of \( x \)) called the normal section of \( M \) at \( x \) in the direction of \( X \), where \( s \) denotes the arc length of \( \gamma \) [1].

**Definition 1.** If each normal section \( \gamma \) of \( M \) is a Frenet curve of osculating order \( d \) then \( M \) is said to have \( d \)-planar normal sections (d-PNS). For every normal sections \( \gamma \) of \( M \) if \( \gamma \) is a \( W \)-curve of rank \( d \) in \( M \) then \( M \) is called weak helical submanifold of order \( d \).
Definition 2. If each d-planar normal section is $\gamma$ a geodesic of $M$ then $M$ is said to have geodesic d-planar normal sections (Gd-PNS). For every geodesic normal sections $\gamma$ of $M$ if $\gamma$ is a $W$-curve of rank $d$ in $M$ then $M$ is called weak geodesic helical submanifold of order $d$.

From now on we consider the Frenet curve of osculating order 3 of $IE^m$.

Proposition 3. Let $\gamma$ be a Frenet curve of $IE^m$ of osculating order 3 then we have

$$\gamma''(s) = \kappa_1 v_2, \quad \gamma'(s) = v_1(s)$$

$$\gamma'''(s) = -\kappa_1^2 v_1 + \kappa_1' v_2 + \kappa_1 \kappa_2 v_3$$

(14)

$$\gamma''''(s) = -3\kappa_1 \kappa_1' v_1 + (-\kappa_1^3 + \kappa_1'' - \kappa_1' \kappa_2^2) v_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') v_3.$$  

(15)

Notation: Let us write

$$N_1(s) = \kappa_1 v_2,$$

$$N_2(s) = \kappa_1' v_2 + \kappa_1 \kappa_2 v_3,$$

$$N_3(s) = (-\kappa_1^3 + \kappa_1'' - \kappa_1' \kappa_2^2) v_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') v_3.$$ 

Corollary 4. $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$ and $\gamma''''(s)$ are linearly dependent if and only if $N_1(s)$, $N_2(s)$ and $N_3(s)$ are linearly dependent.

Theorem 5. Let $\gamma$ be a Frenet curve of $IE^m$ of osculating order 3 then

$$N_3(s) = \left\langle N_3(s), N_1^*(s) \right\rangle N_1^*(s) + \left\langle N_3(s), N_2^*(s) \right\rangle N_2^*(s)$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}, \quad N_2^*(s) = \frac{N_2(s) - \left\langle N_2(s), N_1^*(s) \right\rangle N_1^*(s)}{\|N_2(s) - \left\langle N_2(s), N_1^*(s) \right\rangle N_1^*(s)\|}$$

[3].

Definition 6. Frenet curves ( of osculating order 3 ) are

i) of type weak $AW(2)$ if they satisfy

$$N_3(s) = \left\langle N_3(s), N_2^*(s) \right\rangle N_2^*(s),$$

ii) of type weak $AW(3)$ if they satisfy

$$N_3(s) = \left\langle N_3(s), N_1^*(s) \right\rangle N_1^*(s)$$

[3].

Corollary 7. Let $\gamma$ be a Frenet curve of type weak $AW(2)$. If $\gamma$ is a plane curve then $\kappa_1'''(s) - \kappa_1^3(s) = 0$, and the solution of this differential equation is

$$\kappa_1^1 = \pm \frac{\sqrt{2}}{s + c}, \quad c = \text{Const.}$$

[3].

The curvature vector field of $\gamma$ ( the mean curvature vector field of $\gamma$ ) is defined by

$$h(T,T) = H(s) = \gamma''(s) = \kappa_1(s) v_2(s).$$

(16)

One can use the Frenet equations (15) to compute
\( \gamma''' (s) = (-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) \nu_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2^2) \nu_3 \) \quad (17)

**Definition 8.** Curves are of type \( AW(1) \) if they satisfy
\[
\gamma''' (s) = 0, 
\]
of type \( AW(2) \) if they satisfy
\[
\gamma''' (s) \Lambda \gamma'''' (s) = 0 
\]
and of type \( AW(3) \) if they satisfy
\[
\gamma''' (s) \Lambda \gamma'''' (s) = 0. 
\]

**Proposition 9.** Let \( \gamma \) be a Frenet curve of type \( AW(1) \) if and only if
\[
-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2 = 0 
\]
and
\[
2\kappa_1' \kappa_2 + \kappa_1 \kappa_2^2 = 0. 
\]

**Proof.** Substituting (17) into (18) we get the result.

**Corollary 10.** Let \( \gamma \) be a Frenet curve of type \( AW(1) \).

i) If \( \kappa_1 = 0 \) then \( \gamma \) is a straight line.

ii) If \( \kappa_1 \neq 0, \kappa_2 = 0 \) then \( \kappa_1'' - \kappa_1^3 = 0 \). That is
\[
\kappa_1 = \pm \frac{\sqrt{2}}{s + c}, \, c = \text{Const.} 
\]

[3].

iii) If \( \kappa_1, \kappa_2 \neq 0 \) then by (21) and (22) we obtain
\[
\kappa_2 = \frac{c}{\kappa_1^2}, \quad \kappa_1'' - \kappa_1^3 - \frac{c^2}{\kappa_1^3} = 0. 
\]

Putting \( \kappa_1 = y \) into (23) we get
\[
y'' - y^3 - \frac{c^2}{y^3} = 0. 
\]

Thus solving the differential equation (24) one gets
\[
\int y(s) \frac{2a}{\sqrt{2a^2 - 4c^2 + 4C_1a^2}} da - x - C_2 = 0, 
\]

\[
\int y(s) \frac{2a}{\sqrt{2a^2 - 4c^2 + 4C_1a^2}} da - x - C_2 = 0. 
\]

Using \( \kappa_1 = y, \, \kappa_2 = \frac{c}{\kappa_1^2} \), we get the result.

**Corollary 11.** Every plane curve of \( AW(1) \) type is also of weak \( AW(2) \) type [3].

**Proposition 12.** Let \( \gamma \) be a Frenet curve of type \( AW(2) \) if and only if
\[
-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2 = \delta_1 \kappa_1'. 
\]

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2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' = \delta_1 \kappa_1 \kappa_2. \quad (26)

**Proof.** Substituting (14) and (17) into (19) we get the result.

**Corollary 13.** Let \( \gamma \) be a Frenet curve of type \( AW(2) \).

i) If \( \kappa_1 = 0 \) then \( \gamma \) is a straight line.

ii) If \( \kappa_1 \neq 0, \kappa_2 = 0 \) then by (25) we obtain
\[
\kappa_1'' - \delta_1 \kappa_1' = 0 \quad (27)
\]
Putting \( \kappa_1 = y \) into (27) we get
\[
y'' - y^3 - \delta_1 y' = 0 \quad (28)
\]
Thus solving the differential equation (28) one gets
\[
y = c_1 e^{\frac{\delta_1 + \sqrt{4 + \delta_1^2}}{2}} + c_2 e^{\frac{-\delta_1 + \sqrt{4 + \delta_1^2}}{2}}.
\]
Using \( \kappa_1 = y \) we get the result.

iii) If \( \kappa_1, \kappa_2 \neq 0 \) then by (25) and (26) we obtain
\[
\kappa_1''' + \kappa_1'' (3 \kappa_1' - 3 \delta_1 \kappa_1) + \kappa_1' (-3 \delta_1 \kappa_1' - 6 \kappa_1^3 + 2 \delta_1^2 \kappa_1) + 2 \delta_1 \kappa_1^4 = 0 \quad (29)
\]
Putting \( \kappa_1 = y \) and \( \delta_1 = c \) into (29) we get
\[
y''' + y'' (3 y' - 3 c y) + y' (-3 c y' - 6 y^3 + 2 c^2 y) + 2 c y^4 = 0. \quad (30)
\]
Thus solving the differential equation (30) one gets
\[
y(x) = 0, y(x) = -b(-a) \text{ where } \{-b(-a)^6 e^{(-2c,a)} - b(-a)^3 e^{(-2c,a)} c \frac{d}{d_a} b(-a) \}
+ -b(-a)^3 e^{(-2c,a)} \left( \frac{d^2}{d_a^2} b(-a) \right) + C1 = 0, \}
\]
\[
\{x = a, y(x) = -b(-a) \}, \{x = a, y(x) = -b(-a) \}].
\]
Using \( \kappa_1 = y \) we get the result.

**Proposition 14.** Let \( \gamma \) be a Frenet curve of type \( AW(3) \) if and only if
\[
-\kappa_1^3 + \kappa_1'' - \kappa_2 \kappa_1' = \delta_2 \kappa_1 \quad (31)
\]
\[
2 \kappa_1' \kappa_2 + \kappa_1 \kappa_2' = 0. \quad (32)
\]

**Proof.** Substituting (16) and (17) into (20) we get the result.

**Corollary 15.** Let \( \gamma \) be a Frenet curve of type \( AW(3) \).

i) If \( \kappa_1 = 0 \) then \( \gamma \) is a straight line.

ii) If \( \kappa_1 \neq 0, \kappa_2 = 0 \) then by (31) we obtain
\[
\kappa_1'' - \delta_2 \kappa_1' = 0 \quad (33)
\]
Putting \( \kappa_1 = y \) and \( \delta_2 = c \) into (33) we get
\[
y'' - y^3 - cy = 0 \quad (34)
\]
Thus solving the differential equation (34) one gets
\[
\int \frac{2}{\sqrt{2_a^4 + 4_a^2 c + 4_c C1}} d_a - x - C2 = 0,
\]
\[
\int -\frac{2}{\sqrt{2 - a^4 + 4a^2c + 4C1}} \, d_a - x - C2 = 0.
\]

Using \( \kappa_1 = y \) we get the result.

**iii)** If \( \kappa_1, \kappa_2 \neq 0 \) then by (31) and (32) we obtain

\[
\kappa_2 = \frac{c}{\kappa_1^2}, \quad \kappa_1'' - \kappa_1^3 - \frac{c^2}{\kappa_1^5} - \delta_2 \kappa_1 = 0. \tag{35}
\]

Putting \( \kappa_1 = y \) and \( \delta_2 = d \) into (35) we get

\[
y'' - \frac{c^2}{y^3} - dy = 0. \tag{36}
\]

Thus solving the differential equation (36) one gets

\[
\int -\frac{2}{\sqrt{4C1a^2 + 2a^6 - 4c^2 + 4d_a^4}} \, d_a - x - C2 = 0
\]

Using \( \kappa_1 = y, \ k_2 = \frac{c}{\kappa_1^2} \), we get the result.

**Corollary 16.** Every Frenet curve of weak \( AW(3) \) type is also of \( AW(3) \) type [3].

### 3. Surfaces of \( AW(k) \) Type

In this part we consider surfaces of \( AW(k) \) type. Let us write

\[
H(X) = h(X, X) \tag{37}
\]

\[
\nabla H(X) = (\nabla_X h)(X, X) \tag{38}
\]

\[
J(X) = (\nabla_X \nabla_X h)(X, X) + 3h(A_{h(X, X)}X, X) \tag{39}
\]

so that \( H : T(M) \rightarrow N(M), \nabla H : T(M) \rightarrow N(M) \) and \( J : T(M) \rightarrow N(M) \) are fibre maps whose restriction to each fibre \( T_X(M) \) is a homogeneous polynomial map, \( H \) is of degree 2 and \( \nabla H \) is of degree 3 and \( J \) is of degree 4.

Then

\[
J_1(\nabla_{e_1} \nabla_{e_1} h)(e_1, e_1) + 3h(A_{h(e_1, e_1)}e_1, e_1) \tag{40}
\]

\[
J_2(\nabla_{e_2} \nabla_{e_2} h)(e_2, e_2) + 3h(A_{h(e_2, e_2)}e_2, e_2). \tag{41}
\]

**Definition 17.** [4] Submanifolds are of type \( AW(1) \) if they satisfy

\[
J \equiv 0 \tag{42}
\]

submanifolds are of type \( AW(2) \) if they satisfy

\[
\|\nabla H\|^2 J = \langle J, \nabla H \rangle \nabla H \tag{43}
\]

and of type \( AW(3) \) if they satisfy

\[
\|H\|^2 J = \langle J, H \rangle H. \tag{44}
\]

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**Proposition 18.** [5] Let $M$ be a connected normally flat surfaces in $IE^d$. $e_3$ is parallel to the mean curvature vector $H$ of $M$ such that

$$A_{e_3} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad A_{e_4} = \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix}.$$  

We give the following results.

**Lemma 19.** From the Codazzi equations and using (4), (5) and (45)

$$2\beta w_2^2 (e_2) = -e_2 (\beta) + \mu w_3^2 (e_1)$$

$$\begin{array}{c}
\lambda - \mu w_1^2 (e_1) = e_2 (\lambda) - \beta w_3^2 (e_2) \\
2\beta w_2^2 (e_1) = e_2 (\beta) + \lambda w_3^2 (e_2).
\end{array}$$

**Lemma 20.** If $M \subset IE^d$ is normally flat surfaces then

$$J_1 = \{e_1^2 (\lambda) - \lambda (w_3^4 (e_1))^2 - 2e_1 (\beta)w_3^4 (e_1) - \beta e_1 (w_3^4 (e_1)) - 3w_1^2 (e_1) e_2 (\lambda)
\lambda
+ 3\beta w_2^2 (e_1) w_3^2 (e_2) + 3\lambda (\lambda^2 + \beta^2)\} e_3$$

$$+ \{e_1^2 (\lambda) - \beta (w_3^4 (e_1))^2 + 2e_1 (\lambda)w_3^4 (e_1) + \lambda e_1 (w_3^4 (e_1)) - 3w_1^2 (e_1) e_2 (\beta)
\lambda
- 3\lambda w_2^2 (e_1) w_3^2 (e_2) + 3\beta (\lambda^2 + \beta^2)\} e_4$$

and

$$J_2 = \{e_2^2 (\mu) - \mu (w_3^4 (e_2))^2 + 2e_2 (\beta)w_3^4 (e_2) + \beta e_2 (w_3^4 (e_2)) + 3w_1^2 (e_2) e_1 (\mu)
\mu
+ 3\beta w_2^2 (e_2) w_3^2 (e_1) + 3\mu (\mu^2 + \beta^2)\} e_3$$

$$+ \{-e_2^2 (\beta) + \beta (w_3^4 (e_2))^2 + 2e_2 (\mu)w_3^4 (e_2) + \mu e_2 (w_3^4 (e_2)) - 3w_1^2 (e_2) e_1 (\beta)
\beta
+ 3\mu w_2^2 (e_2) w_3^2 (e_1) - 3\beta (\mu^2 + \beta^2)\} e_4.$$  

**Proof.** Substituting (4), (5), (6), (8) and (45) into (40) and (41) we get the result.

**Proposition 21.** Let $M \subset IE^d$ be a normally flat surfaces. If $M$ is $AW(1)$ type then $J_1 = 0$ and $J_2 = 0$. That is

$$e_1^2 (\lambda) - (\lambda + 2\mu)(w_3^4 (e_1))^2 + 4\beta w_3^4 (e_1) w_3^2 (e_2) - \beta e_1 (w_3^4 (e_1))$$

$$- 3(\lambda - \mu)(w_3^4 (e_1))^2 + 3\lambda (\lambda^2 + \beta^2) = 0,$$

$$2e_1 (\beta)w_3^4 (e_1) + (\lambda + \mu) e_1 (w_3^4 (e_1)) + (\lambda - 3\mu) w_3^4 (e_1) w_3^2 (e_2)$$

$$- \beta\{2(w_3^4 (e_1))^2 - 4(w_3^2 (e_1))^2 + 2e_1 (w_3^2 (e_1)) + 6(w_3^4 (e_1))^2 - 3(\lambda^2 + \beta^2)\} = 0,$$

$$e_2^2 (\mu) - (\mu + 2\lambda)(w_3^4 (e_2))^2 + 4\beta w_3^4 (e_2) w_3^2 (e_1) + \beta e_2 (w_3^4 (e_2))$$

$$+ 3(\lambda - \mu)(w_3^4 (e_2))^2 + 3\mu (\mu^2 + \beta^2) = 0,$$

$$2e_2 (\mu) w_3^4 (e_2) + (\lambda + \mu) e_2 (w_3^4 (e_2)) + (3\lambda - \mu) w_3^4 (e_2) w_3^2 (e_1)$$

$$- \beta\{-2(w_3^4 (e_2))^2 + 4(w_3^2 (e_1))^2 + 2e_2 (w_3^2 (e_1)) - 6(w_3^4 (e_2))^2 + 3(\mu^2 + \beta^2)\} = 0.$$
Proof. Substituting (46), (47), (48), (49) into (50) and (51) and from \(AW(I)\) type definition we get the result.

**Proposition 22.** Let \(M\) be a normally flat and has got constant principal curvature submanifold. Then
\[
J_1 = \{-\lambda (w_3^4(e_1)) - \beta e_1(w_3^4(e_1)) + 3\beta w_1^4(e_1)w_3^4(e_2) + 3\lambda (\lambda^2 + \beta^2)\} e_3 \\
+ \{-\beta(w_3^4(e_1))^2 + \lambda e_1(w_3^4(e_1)) - 3\lambda w_1^4(e_1)w_3^4(e_2) + 3\beta(\lambda^2 + \beta^2)\} e_4,
\]
\[
J_2 = \{-\lambda (w_3^4(e_2)) + \beta e_2(w_3^4(e_2)) + 3\beta w_1^4(e_2)w_3^4(e_1) + 3\lambda (\lambda^2 + \beta^2)\} e_3 \\
+ \{-\beta(w_3^4(e_2))^2 + \lambda e_1(w_3^4(e_2)) + 3\lambda w_1^4(e_2)w_3^4(e_1) - 3\beta(\lambda^2 + \beta^2)\} e_4.
\]

**Lemma 23.** Let \(M\) be a normally flat and has got constant principal curvature submanifold of \(AW(I)\) type

i) If \(\lambda = \beta = 0\) then \(M\) is a plane,

ii) If \(\lambda = -\beta\) then \(M\) has got vanishing Gaussian curvature \((K = 0)\), mean curvature \(H = \lambda\) or \(e_2(w_3^4(e_2)) = -3w_3^7(e_2)w_3^4(e_1)\).

iii) If \(\lambda = \beta\) then \(M\) has got vanishing Gaussian curvature \((K = 0)\), mean curvature \(H = \lambda\) or \(e_2(w_3^4(e_2)) = -3w_3^7(e_2)w_3^4(e_1)\).

**Theorem 24.** [3] Let \(\gamma\) be a Frenet curve of order 3 and of type \(AW(k)\) then the cylinder over \(\gamma\) is of type \(AW(k)\), where \(k=1,2,3\).

**Example 25.** Let \(\gamma(s) = \left(\frac{s}{0} \cos(P_1(t))dt, \frac{s}{0} \sin(P_1(t))dt\right)\) be a polinomial spiral with \(\kappa_\gamma(s) = P_k'(t) = \pm \frac{\sqrt{2}}{s+c}, c=\text{Const}\). The Riemannian product of \(\gamma(s)\) with the helicoid \(x(w, t) = (w\cos t, w\sin t, at)\) is of \(AW(I)\) type.

**Example 26.** We define helical cylinder \(H^2\) embedded in \(IE^4\) by \(x(u, v) = (u, acosv, asinv, bv) : a, b \in IR\) and we show that \(H^2\) is of type \(AW(3)\). For \(p=(u, acosv, asinv, bv)\)

\[T_p(H^2)\text{ is spanned by }x_u=(1, 0, 0, 0)\]
\[x_v=(0, -asinv, acosv, b)\]

and \(N_p(H^2)\text{ is spanned by }n_1=(0, \cos v, \sin v, 0)\]
\[n_2=(0, -\frac{b}{a}\sin v, -\frac{b}{a}\cos v, 1)\].

We have the orthonormal frame \(X, Y, v_j, v_2\) where
\[
X = x_u = (1,0,0,0)
\]
\[
Y = \frac{x_v}{\|x_v\|} = \frac{1}{\sqrt{a^2 + b^2}} (0, -a \sin v, a \cos v, b)
\]
\[
v_1 = \frac{n_1}{\|n_1\|} = (0, \cos v, \sin v, 0)
\]
\[
v_2 = \frac{n_2}{\|n_2\|} = \frac{a}{\sqrt{a^2 + b^2}} (0, b \sin v, -b \cos v, 1).
\]

Differentiating these we have
\[
\tilde{\nabla}_X X = \tilde{\nabla}_Y Y = \tilde{\nabla}_Y Y = 0,
\]
\[
\tilde{\nabla}_v v_1 = \tilde{\nabla}_v v_2 = 0,
\]
\[
\tilde{\nabla}_v v_1 = \frac{a}{a^2 + b^2} Y - \frac{b}{a^2 + b^2} v_2, 
\]
\[
\tilde{\nabla}_v v_2 = \frac{b}{a^2 + b^2} v_1.
\]

Combining these with (1) and (2) we get
\[

\nabla_X X = \nabla_Y Y = \nabla_Y Y = 0,
\]
\[
h(X, X) = h(X, Y) = h(Y, X) = 0,
\]
\[
h(Y, Y) = \frac{-a}{a^2 + b^2} v_1
\]
\[
A_{v_1} X = A_{v_2} X = A_{v_2} Y = 0,
\]
\[
A_{v_1} Y = \frac{-a}{a^2 + b^2} Y
\]
\[
D_{v_1} v_1 = D_{v_2} v_2 = 0,
\]
\[
D_{v_1} v_1 = \frac{-b}{a^2 + b^2} v_2, 
\]
\[
D_{v_2} v_2 = \frac{b}{a^2 + b^2} v_1.
\]

Substituting (6), (8), (54), (55), (56) and (57) into (40) and (41) we have
\[
J(X) = J = 0,
\]
\[
J(Y) = J_2 = \frac{a(b^2 - 3a^2)}{(a^2 + b^2)^3} v_1.
\]

Substituting (37) and (58) into (44) we get the result.

**Example 27.** We define surfaces embedded in \( IE^4 \) by
\[
x(u, v) = (u, v, u \cos v, u \sin v)
\]
and we show that surfaces is of type \( AW(3) \).

After some calculations we get
\[
J(X) = J_1 = 0,
\]
\[
J(Y) = J_2 = \frac{-\sqrt{2}u}{(1 + u^2)^3} v_1.
\]

Substituting (37) and (59) into (44) we get the result.

**Example 28.** We define surfaces embedded in \( IE^4 \) by
\[
x(u, v) = \{(ucosv, usinv, \cos b, \sin b) : b \in IR\}
\]
and we show that surfaces is of type \( AW(3) \).

After some calculations we get
\[
J(X) = J_1 = 0,
\]
\[
J(Y) = J_2 = \frac{b^2(2u^2b^2 + 8u^2 - 3b^4)}{(u^2 + b^2)^3} v_1.
\]

Substituting (37) and (60) into (44) we get the result.

**Example 29.** We define a Mobius band \( M^2 \) embedded in \( IE^4 \) by
\[
x(u, v) = (cos u, sin u, vcos \frac{u}{2}, vsin \frac{u}{2})
\]

60
and we show that $M^2$ is of type $AW(3)$.

After some calculations we get
\[ J(X) = J_1 = \frac{-144}{(4 + \nu^2)^3} \nu_1, \quad J(Y) = J_2 = 0. \tag{61} \]

Substituting (37) and (61) into (44) we get the result.

REFERENCES